COGNOME ______ NOME ______ N. Matricola ______

Calcolo Numerico - II prova intermedia

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Esercizio 1

Approssimare

$$\int_{-2}^{1} \frac{x+1}{x^2+2} dx$$

usando il metodo dei trapezi con 6 sottointervalli. Dare una stima a posteriori dell'errore.

x_m	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
$f(x_m)$	$\frac{-1}{6}$	$\frac{-\frac{1}{2}}{\frac{9}{4}+2} = \frac{-2}{17}$	0	$\frac{\frac{1}{2}}{\frac{1}{4}+2} = \frac{2}{9}$	$\frac{1}{2}$	$\frac{\frac{3}{2}}{\frac{1}{4}+2} = \frac{2}{3}$	$\frac{2}{3}$

6 sottointervalli $\leadsto$ $H = \frac{1-(-2)}{6} = \frac{1}{2}$ $\qquad x_m = -2 + mH \quad m = 0,1,\dots,6$

$$I_6^T(f) = \frac{1}{2}\left[\frac{1}{2}\,\frac{-1}{6} + \frac{-2}{17} + 0 + \frac{2}{9} + \frac{1}{2} + \frac{2}{3} + \frac{1}{2}\,\frac{2}{3}\right]$$

$$= \frac{1}{2}\left[\frac{-1}{12} - \frac{2}{17} + \frac{2}{9} + \frac{1}{2} + \frac{2}{3} + \frac{1}{3}\right] = 0.760621$$

Per stimare l'errore mi serve calcolare l'integrale approssimato con $\frac{6}{2} = 3$ sottointervalli

$$I_3^T(f) = 1\left[\frac{1}{2}\,\frac{-1}{6} + 0 + \frac{1}{2} + \frac{1}{2}\,\frac{2}{3}\right] = \frac{-1}{12} + \frac{1}{2} + \frac{1}{3} = \frac{-1+6+4}{12} = \frac{9}{12} = 0.75$$

$$|E_6^T(f)| = \left|\int_{-2}^{1} \frac{x+1}{x^2+2} dx - I_6^T(f)\right| \simeq \frac{|I_6^T(f) - I_3^T(f)|}{2^2 - 1}$$

$$= \frac{1}{3}\,|0.760621 - 0.75| = \frac{1}{3}\,0.010621 = 0.003540$$

Esercizio 2

Dato il problema di Cauchy

$$\begin{cases} y' = \dfrac{2y}{t+3} & t \in [1,2] \\ y(1) = 1 \end{cases}$$

approssimare la soluzione usando il metodo di Crank-Nicolson

$$u_{n+1} = u_n + \frac{h}{2}(f_n + f_{n+1})$$

con passo $h = 0.25$.

$$u_{m+1} = u_m + \frac{h}{2}\left[\frac{2u_m}{t_m+3} + \frac{2u_{m+1}}{t_{m+1}+3}\right]$$

$$\left(1 - \frac{h}{t_{m+1}+3}\right)u_{m+1} = \left(1 + \frac{h}{t_m+3}\right)u_m$$

$$u_{m+1} = \frac{1+\frac{h}{t_m+3}}{1-\frac{h}{t_{m+1}+3}}\,u_m = \frac{\frac{t_m+3+h}{t_m+3}}{\frac{t_{m+1}+3-h}{t_{m+1}+3}}\,u_m =$$

$$t_m + h = t_{m+1}$$

$$t_{m+1} - h = t_m$$

$$= \left(\frac{t_{m+1}+3}{t_m+3}\right)^2 u_m$$

$$u_0 = 1$$

$$u_1 = \left(\frac{1.25+3}{1+3}\right)^2 = \left(\frac{4.25}{4}\right)^2$$

$$u_2 = \left(\frac{1.5+3}{1.25+3}\right)^2\left(\frac{4.25}{4}\right)^2 = \left(\frac{4.5}{4}\right)^2$$

$$u_3 = \left(\frac{1.75+3}{1.5+3}\right)^2\left(\frac{4.5}{4}\right)^2 = \left(\frac{4.75}{4}\right)^2$$

$$u_4 = \left(\frac{2+3}{1.75+3}\right)^2\left(\frac{4.75}{4}\right)^2 = \left(\frac{5}{4}\right)^2$$

Esercizio 3

Verificare che il metodo per l'approssimazione della soluzione di un problema di Cauchy a due passi

$$u_{n+1} = \frac{3}{2}u_n - \frac{1}{2}u_{n-1} + \frac{h}{8}(5f_{n+1} - f_{n-1})$$

ha ordine di consistenza 2.

$$\tau_{m+1}(h) = \frac{1}{h}\left[y(t_{m+1}) - \frac{3}{2}y(t_m) + \frac{1}{2}y(t_{m-1}) - \frac{h}{8}\left(5y'(t_{m+1}) - y'(t_{m-1})\right)\right]$$

$$= \frac{1}{h}\Big[y(t_m) + h y'(t_m) + \frac{h^2}{2}y''(t_m) + \frac{h^3}{6}y'''(t_m) + O(h^4)$$

$$-\frac{3}{2}y(t_m)$$

$$+\frac{1}{2}\left(y(t_m) - h y'(t_m) + \frac{h^2}{2}y''(t_m) - \frac{h^3}{6}y'''(t_m)\right)$$

$$-h\frac{5}{8}\left(y'(t_m) + h y''(t_m) + \frac{h^2}{2}y'''(t_m)\right)$$

$$+\frac{h}{8}\left(y'(t_m) - h y''(t_m) + \frac{h^2}{2}y'''(t_m)\right)\Big]$$

$$= \frac{1}{h}\Big[\left(\cancel{1 - \frac{3}{2} + \frac{1}{2}}\right)y(t_m) + h\left(\cancel{1 - \frac{1}{2} - \frac{5}{8} + \frac{1}{8}}\right)y'(t_m) + h^2\left(\cancel{\frac{1}{2} + \frac{1}{4} - \frac{5}{8} - \frac{1}{8}}\right)y''(t_m)$$

$$+ h^3\left(\frac{1}{6} - \frac{1}{12} - \frac{5}{16} + \frac{1}{16}\right)y'''(t_m) + O(h^4)\Big]$$

$$= h^2\left(\frac{1}{12} - \frac{1}{4}\right)y'''(t_m) + O(h^3) = h^2\,\frac{-1}{6}\,y'''(t_m) + O(h^3)$$

Quindi $\tau(h) = \max\limits_{2 \le m \le N} |\tau_m(h)| = O(h^2)$

Esercizio 4

Scrivere una funzione di Matlab che implementi il seguente metodo di Runge-Kutta a tre stadi

$$u_{n+1} = u_n + \frac{h}{4}(K_1 + 3K_3)$$

$$K_1 = f(t_n, u_n)$$

$$K_2 = f\left(t_n + \frac{h}{3}, u_n + \frac{h}{3}K_1\right)$$

$$K_3 = f\left(t_n + \frac{2h}{3}, u_n + \frac{2h}{3}K_2\right).$$

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function   [t,u]=RK3( fun, tø, yø, T, N)
h= T/N;
t=[tø:h:tø+T];
u(1)=yø;
for   n=1:N
    K1 = feval( fun, t(n), u(n));
    K2= feval( fun, t(n)+ h/3 , u(n)+ h/3 *K1);
    K3= feval( fun, t(n)+2*h/3, u(n)+2*h/3*K2);
    u(n+1)= u(n)+ h/4 *(K1 +3*K3);
end.
```